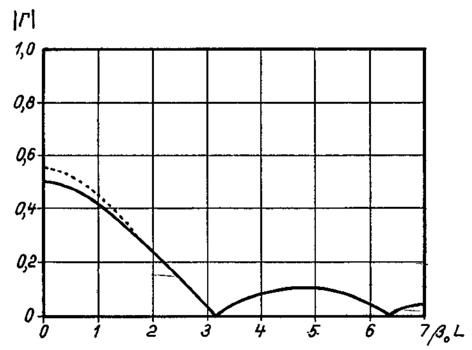


Fig. 2. $|\Gamma|$ versus $\beta_0 L$ at an impedance ratio of 3:1.

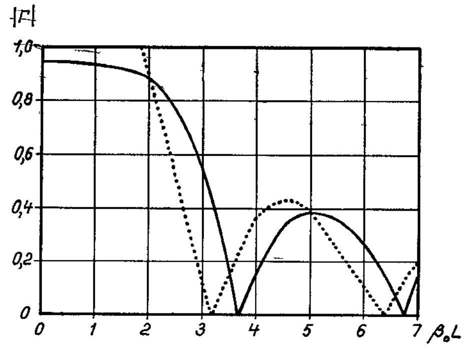


Fig. 3. $|\Gamma|$ versus $\beta_0 L$ at an impedance ratio of 54:1.

tance $Y_p(x)$ of the line are small, as is often the case at low frequencies. Also in this case we have to solve a Riccati equation, and the solution is given by (3) if the exponential term is omitted and the symbols are exchanged according to the scheme

$$\begin{cases} \Gamma(x) \to Y(x) \\ \Gamma(0) \to Y(0) \\ f_1(x) \to Z_s(x) \\ f_2(x) \to Y_p(x). \end{cases} \tag{7}$$

The solution for $Y(x)$ has turned out to be particularly useful, for example in calculations of the propagation of plane waves in lossy, stratified media at low frequencies.

## IV. A Numerical Example

In order to give an idea of the usefulness of (3) in a practical case, the magnitude of the reflection coefficient $\Gamma(x)$ of a matched exponential line has been computed for two different impedance ratios. This particular line has been chosen since it is also possible to derive an exact expression for its reflection coefficient which can be compared with (3). The characteristic admittance of the exponential line is supposed to vary as $Y_c(x) = Y_0 \cdot \exp(2\delta x)$, while its propagation factor is $\gamma = j\beta_0$. The two values of $\delta L$ that have been used for $\delta x$ ($\delta$ is a constant and $L$ is the length of the line) are $\delta L = 0.55$ and $\delta L = 2$, corresponding to impedance ratios of 3:1 and 54:1, respectively. (Since this is only an illustrative example, no attention is payed here to the physical realizability of such lines.) The highest order term used is $K_5$. At the lowest values of $\beta_0 L$, the equation for $Y(x)$ has been used instead of (3) to obtain better convergence.

The results, which are shown in Figs. 2 and 3, are rather striking. The curves one gets with the method presented here (solid curves) coincide within drawing accuracy with the exact ones for all values of $\beta_0 L$ at both impedance ratios. The curves obtained when the problem is solved in the conventional way (dotted curves), on the other hand, show serious disagreement particularly for the high impedance ratio and at low values of $\beta_0 L$.

## V. Conclusion

In this short paper equations are presented for the electrical properties of a nonuniform line. They are given in series form and are valid also for lossy lines connected to arbitrary loads. The equations may be applied to all kinds of single-mode transmission lines; for instance, coaxial lines, strip lines, and waveguides. As a consequence they can be utilized in the design of many microwave components containing nonuniform line sections, like resonators, filters, tapered transitions, etc. In cases where one has the choice, nonuniform line sections often have advantages over uniform ones. The usefulness of the equations derived here is, however, not limited to transmission lines only. Due to the analogy between the free propagation of plane waves in a medium and waves on transmission lines, the results may also be used in the design of certain types of absorbing materials or in the study of propagation of plane waves in a stratified atmosphere, to take only two examples.

The well-known fact that the Riccati equation can be transformed by a simple mathematical operation into a one-dimensional wave equation indicates that the equations may be applied in other fields of physics as well. Thus, for example, it may well be expected that the solution described in this short paper could be used with benefit in such fields as acoustics, optics, and quantum mechanics.

## Acknowledgment

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# The Lowest Order Mode and the Quasi-TEM Mode in a Ferrite-Filled Coaxial Line or Resonator

INGO WOLFF

***Abstract*—The field distribution of the mode in a ferrite-filled coaxial cavity, which converges towards the TEM mode in the isotropic case, is discussed.**

During the last few years there has been a discussion between M. M. Weiner and M. E. Brodwin and D. A. Miller about the "lowest order mode" and an approximate theory for this mode, called the quasi-TEM mode, in a ferrite-filled coaxial line [1]–[3]. The author has studied the behavior of all modes in a ferrite-filled coaxial cavity [4], [5] and would like to give some detailed results for the "lowest order mode" and the correct conditions for approximating it by the Suhl and Walker approximation [7] of a quasi-TEM mode.

Basically, there are three different kinds of modes in a ferrite-filled cavity.

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The author is with the Institut für Hochfrequenztechnik, Technische Hochschule Aachen, 51 Aachen, Germany.

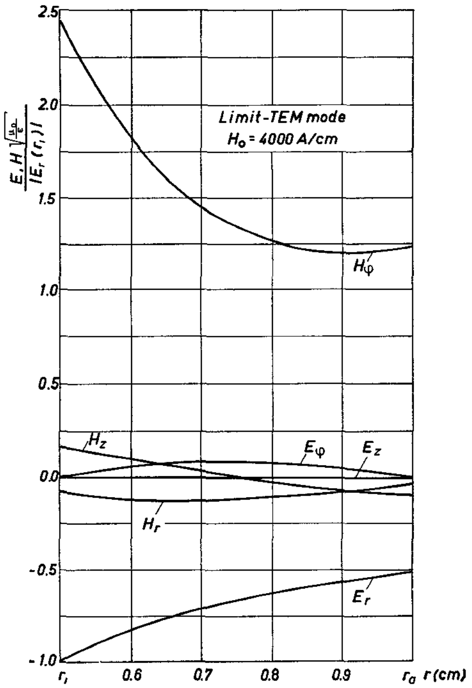


Fig. 1. The dependence of the field components on the radial coordinate for the limit-TEM mode in a ferrite-filled coaxial cavity. Material R5, $r_a=1.0$ cm, $r_i=0.5$ cm, $l=0.5$ cm, $H_0=4$ kA/cm.

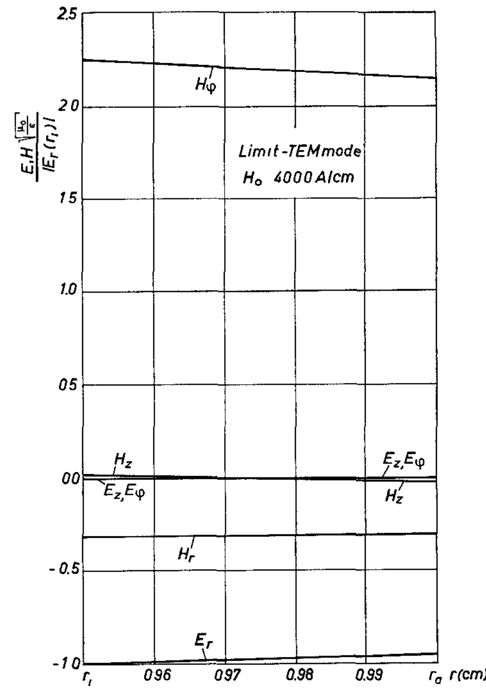


Fig. 2. The dependence of the field components on the radial coordinate for the limit-TEM mode in a ferrite-filled coaxial cavity. Material R5, $r_a=1.0$ cm, $r_i=0.95$ cm, $l=0.5$ cm, $H_0=4$ kA/cm.

1) The TM mode, which is independent of the longitudinal coordinate ($z$ axis).

2) The HE or EH modes, which converge towards the TE or TM modes in the isotropic limit. The isotropic limit is given in the case of very high magnetic bias field ($H_0 \to \infty$) or vanishing magnetic bias field [4], [5].

3) There is a third mode which converges towards the TEM mode in the isotropic limit: The so-called "lowest order mode."

It is not proper to call it the "lowest order mode," because in a coaxial cavity the lowest order mode may be the mode which converges towards the TEM mode (for $H_0 \to \infty$) or the $z$-independent TM mode depending upon the dimensions chosen. We, therefore, would prefer to call the mode, which converges towards the TEM mode, the "limit-TEM mode."

This mode must satisfy the following conditions.

1) The field distribution of the limit-TEM mode must be independent of the azimuthal angle.

2) The eigenvalues of the differential equations

$$\Delta_t E_z + s_{1,2}^2 E_z = 0$$
$$\Delta_t H_z + s_{1,2}^2 H_z = 0$$

must converge towards zero in the isotropic limit. They also become zero if the frequency approaches zero.

3) The field of the limit-TEM mode has all six field components, in particular the $E_z$ and $H_z$ components.

4) The $E_z$ and $H_z$ components as well as the $E_\phi$ and $H_r$ components must vanish in the isotropic limit.

As calculations on a digital computer show, there exist solutions which satisfy the above mentioned conditions. For these particular solutions the eigenvalue $s_1^2$ is positive, and $s_2^2$ is negative for all values of magnetic bias field. For infinite magnetic bias field both eigenvalues converge towards zero. Fig. 1 shows the dependence of the field components on the radial coordinate. It may be seen that for a bias field ($H_0=4$ kA/cm) above gyromagnetic resonance and for the dimensions chosen ($r_i=0.5$ cm, $r_a=1.0$ cm, $l=0.5$ cm, where $l$ is the length of the cavity) the $E_\phi$ and $E_z$ components are small compared to the $E_r$ component, and the $H_r$ and $H_z$ components are small compared to the $H_\phi$ component. The $E_r$ and $H_\phi$ components approximately decrease as $1/r$. The $E_z$ component is also small compared to the $E_\phi$ component, but the magnitude of the $H_z$ component is approximately that of the $H_r$ component. In case of magnetic bias fields below gyromagnetic resonance, the $E_r$ component is no longer large compared to that of the $E_\phi$ component, but both components are much larger than the $E_z$ component [6]. Below gyromagnetic resonance the $H_\phi$ component is still large compared to the $H_r$ and $H_z$ component. Field distributions of the elliptically polarized fields are given in [4]–[6] by the author.

In their communication M. E. Brodwin and D. A. Miller [2] stated that the conditions for approximating the "lowest order mode" by the Suhl and Walker approximation of a quasi-TEM mode are

$$|s_1|(r_a - r_i) \ll 1$$
$$|s_2|(r_a - r_i) \ll 1.$$

In contradiction Weiner [1], [3] said that this condition is not sufficient and that the necessary and sufficient conditions should be

$$|s_1| r_a \ll 1$$
$$|s_1| r_i \ll 1$$
$$|s_2| r_a \ll 1$$
$$|s_2| r_i \ll 1.$$

In order to prove the conditions, resonators with small and large differences between the inner and outer radii were calculated. Fig. 2 shows the field distribution for a bias field of $H_0=4$ kA/cm and the dimensions $r_i=0.95$ cm, $r_a=1.0$ cm, $l=0.5$ cm. As may be seen from Fig. 2, the $E_\phi$ and the $E_z$ components are small compared to the $E_r$ component. The magnetic-field strength has an $H_\phi$ component and an $H_r$ component. The $H_z$ component is very small. Therefore, the field distribution may be approximated by a quasi-TEM mode. However, the quantities $|s_1 r_a|$ and $|s_2 r_a|$ are greater than 1.0 in the case considered here ($s_1 r_a = 2.28$, $s_2 r_a = j2.15$) so that the conditions given by Weiner [1], [3], which should be satisfied when using the quasi-TEM mode instead of the limit-TEM mode, are not satisfied. If a magnetic bias field of 8 kA/cm and the resonator with the dimen-

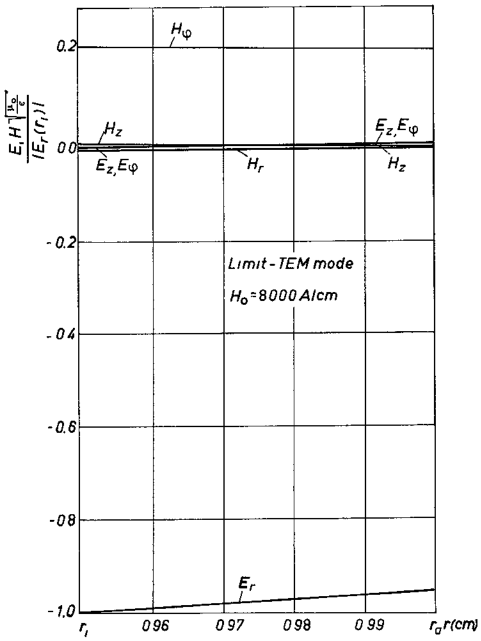


Fig. 3. The dependence of the field components on the radial coordinate for the limit-TEM mode in a ferrite-filled coaxial cavity. Material R5, $r_a = 1.0$ cm, $r_i = 0.95$ cm, $l = 0.5$ cm, $H_0 = 8$ kA/cm.

TABLE I

DIFFERENCES BETWEEN THE NUMERICALLY CALCULATED RESONANCE FREQUENCIES OF A FERRITE-FILLED COAXIAL CAVITY FOR THE LIMIT-TEM MODE AND QUASI-TEM MODE AS A FUNCTION OF THE RESONATOR DIMENSIONS

| $r_i$ (cm) | $r_a$ (cm) | $l$ (cm) | $H_0$ (kA/cm) | $s_1 r_a$ | $s_2 r_a$ | $\mu_{eff}$ | Difference (%) |
|---|---|---|---|---|---|---|---|
| 0.1 | 1.0 | 0.5 | 4.0 | 2.056 | $j2.309$ | 1.326 | 1.240 |
| 0.5 | 1.0 | 0.5 | 4.0 | 2.178 | $j2.226$ | 1.327 | 0.595 |
| 0.9 | 1.0 | 0.5 | 4.0 | 2.280 | $j2.151$ | 1.328 | 0.029 |
| 0.93 | 1.0 | 0.5 | 4.0 | 2.283 | $j2.149$ | 1.328 | 0.017 |
| 0.95 | 1.0 | 0.5 | 4.0 | 2.284 | $j2.148$ | 1.328 | 0.011 |
| 0.98 | 1.0 | 0.5 | 4.0 | 2.285 | $j2.148$ | 1.328 | 0.006 |
| 0.1 | 1.0 | 0.5 | 0.4 | 3.125 | $j5.047$ | 0.746 | 18.210 |
| 0.5 | 1.0 | 0.5 | 0.4 | 3.872 | $j4.460$ | 0.779 | 8.520 |
| 0.9 | 1.0 | 0.5 | 0.4 | 4.546 | $j3.731$ | 0.807 | 0.275 |
| 0.93 | 1.0 | 0.5 | 0.4 | 4.559 | $j3.713$ | 0.807 | 0.151 |
| 0.95 | 1.0 | 0.5 | 0.4 | 4.566 | $j3.710$ | 0.807 | 0.093 |
| 0.98 | 1.0 | 0.5 | 0.4 | 4.571 | $j3.698$ | 0.807 | 0.040 |

sions given above is used, the field distribution in the first approximation becomes the field of a TEM mode in a coaxial resonator, Fig. 3. The $E_r$ and the $H_\phi$ components are at least twenty times larger than the other field components. In this case $s_1 r_a = 1.156$ and $s_2 r_a = j1.41$. For $H_0 = 4$ kA/cm, the difference between the propagation constant (or the resonance frequency) calculated from the exact determinant and that calculated from the Suhl and Walker approximation of the quasi-TEM mode is small (see Table I). This difference becomes smaller if the difference between the inner and the outer radii gets smaller. This is still true for smaller magnetic bias field (see Table I, $H_0 = 400$ A/cm). It means that the conditions given by Weiner [1], [3] are by no means the most general ones.

Weiner [3] has mentioned, that for a small difference between the inner and outer radii the determinantal equation is identical zero. This is absolutely wrong. In its correct form this sentence should be: If the difference between the inner and the outer radii of the ferrite-filled coaxial line is small

$$|s_1| (r_a - r_i) \ll 1$$
$$|s_2| (r_a - r_i) \ll 1$$

which means

$$|s_1| r_a = |s_1| r_i + \epsilon$$
$$|s_2| r_a = |s_2| r_i + \epsilon$$

with $\epsilon \ll 1$ and the Bessel functions are approximated by [8]

$$J_0(|s_{1,2}r_i| + \epsilon) \approx J_0(|s_{1,2}r_i|) - \epsilon J_1(|s_{1,2}r_i|)$$
$$N_0(|s_{1,2}r_i| + \epsilon) \approx N_0(|s_{1,2}r_i|) - \epsilon N_1(|s_{1,2}r_i|)$$
$$I_0(|s_{1,2}r_i| + \epsilon) \approx I_0(|s_{1,2}r_i|) + \epsilon I_1(|s_{1,2}r_i|)$$
$$K_0(|s_{1,2}r_i| + \epsilon) \approx K_0(|s_{1,2}r_i|) - \epsilon K_1(|s_{1,2}r_i|)$$

the determinantal equation vanishes to a first approximation. It means, the value of the determinantal equation is small, but by no means zero. It further means that a second-order approximation has to be done to find that propagation constant, for which the determinantal equation changes its sign, thereby well defining the zero of the determinant. Weiner seems not to have done this. If it is done, it can be shown that the determinantal equation only vanishes exactly, if

$$\beta = \omega\sqrt{\epsilon_0 \epsilon_r \mu_{eff}}.$$

Therefore, the conditions given by Weiner are by no means correct and the conditions given by Brodwin and Miller [2] are sufficient conditions. Mueller and Rosenbaum repeat the conditions given by Weiner [9]. Lewin [10] states that there can exist a large difference between the exact solution and the quasi-TEM solution, but he only gives an example with $r_a/r_i = 1.5$, which does not satisfy the above-given conditions and, therefore, is in no contradiction to the conclusions made here.

Summarizing, the conditions given by Weiner are not correct, and the conditions given by Brodwin and Miller are sufficient conditions for approximating the limit-TEM mode by the Suhl and Walker approximation.

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